

Mixing by a turbulent wake of a uniform temperature gradient in the approach flow

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The mean temperature distribution is analyzed in the turbulent wake of a circular cylinder when there is a linear temperature gradient flow upstream of the cylinder. Using the same variable eddy diffusivity derived from published temperature measurements in the wakes of heated cylinders, it is shown that the mean temperature distribution has a self-similar form which agrees with experimental results of Alexopoulos and Keffer. But, the location and magnitude of the maximum temperature *perturbations* are in error by 25% and 50%, respectively. The reasons for the limitations of the eddy diffusivity model in wake flows are discussed. The variance of temperature fluctuations $(\overline{\theta'^2})$ in a linear temperature gradient mixed by homogeneous grid turbulence are then analyzed on the basis of Durbin's theory. It is shown that the measurements of $(\overline{\theta'^2})$ are more plausibly fitted by the curve $(\overline{\theta'^2}) \propto x^{1/2}$ than by Sullivan's analysis of the same data.

I. INTRODUCTION

There is considerable interest in understanding the mixing and dispersion of passive contaminants in turbulent wakes: for example, in nuclear reactor safety studies, the mean and fluctuating temperature distributions behind obstructions in fuel elements need to be estimated.¹ There have been various calculations, based on eddy diffusivity or more complex models, of the temperature distribution in the wake of a single heated cylinder.^{2,3} In this paper, we consider the different, but related problem, of the mixing of a linear temperature profile in the wake of a nonheated cylinder.

In Secs. II and III we analyze, respectively, the mean and fluctuating temperature fields measured in the experiments of Alexopoulos and Keffer.⁴ The analysis of the mean temperature is based on the eddy diffusion hypothesis, using standard eddy diffusivity values obtained for the heated cylinder.³ It is applied to the main experiments of Ref. 4 in which a linear temperature profile was eroded by the turbulent wake of a nonheated cylinder. This is an interesting test of the eddy diffusion hypothesis. The essential assumption in eddy diffusivity models is that the diffusivity is independent of the temperature field. Even in the more complex " $k - \epsilon$ " turbulent models,² where the diffusivity is defined in terms of the turbulence by two equations, the diffusivity is independent of the temperature distribution because the same values of eddy diffusivity should apply in both cases, that is, heated and unheated cylinders.

The analysis of the fluctuating temperature (temperature variance) is based on an extension to the Lagrangian theory of dispersion suggested by Durbin.⁵ In this case, consideration is restricted primarily to Alexopoulos and Keffer's experiment in which a linear temperature profile was mixed by homogeneous grid turbulence in the absence of any cylinder.

Our analysis highlights many shortcomings, experi-

mental and theoretical, in the present understanding of turbulent dispersion and mixing.

II. ANALYSIS OF THE MEAN FIELD

Consider a circular cylinder to be placed across a uniform flow. A grid of heated wires is placed upstream and produces a mean temperature gradient across the flow (Fig. 1). It is assumed that the turbulence produced by the grid is negligible. The essential feature of the flow is that the turbulence produced by the cylinder mixes fluid across the wake. This tends to make the temperature uniform within the wake and to produce temperature fluctuations.

A. Governing equations

Let the mean temperature be $\Theta(x, y)$. The mean velocity is $(U + u, v)$, U being the uniform upstream velocity and (u, v) the wake deficit. We consider only the region far downstream of the cylinder, where $|u|, |v| \ll U$. Thus, we ignore u, v and treat the mean flow as uniform and equal to $(U, 0)$. Turbulent transport transverse to the flow is described by an eddy diffusivity $K_e(x, y)$: the shortcomings of the eddy diffusion concept will be discussed later.

With these assumptions, the equation governing Θ is

$$U \frac{\partial \Theta}{\partial x} = \frac{\partial}{\partial y} \left(K_e \frac{\partial \Theta}{\partial y} \right). \quad (1a)$$

If the mean temperature is expressed as the sum of the initial temperature profile $T(y)$ and a perturbation θ , so that

$$\Theta = T(y) + \theta(x, y),$$

then (1a) becomes

$$U \frac{\partial \theta}{\partial x} - \frac{\partial}{\partial y} \left(K_e \frac{\partial \theta}{\partial y} \right) = \frac{\partial T}{\partial y} \frac{\partial K_e}{\partial y}. \quad (1b)$$

Since $\theta = 0$ at $x = 0$ and as $y \rightarrow \pm\infty$, this shows that the temperature perturbation is effectively produced by the

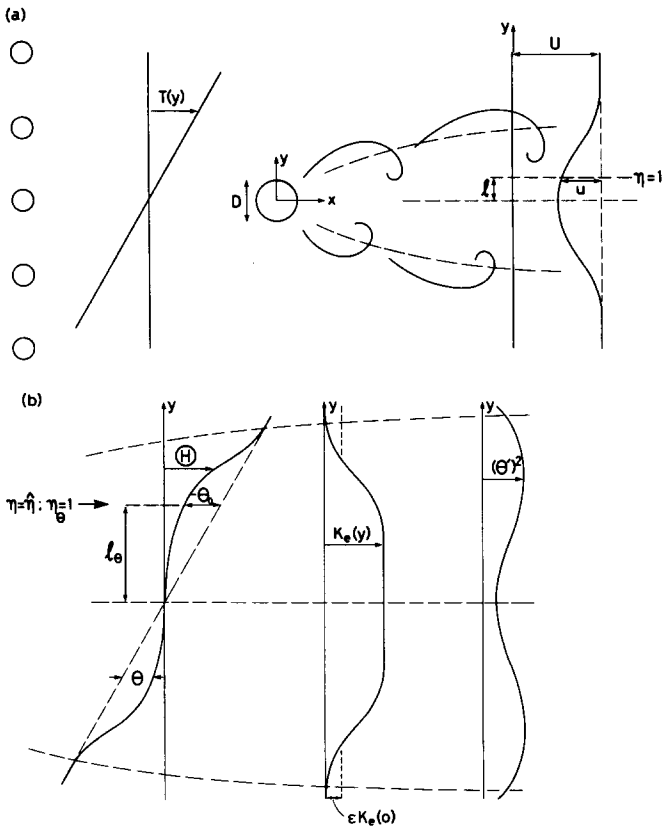


FIG. 1. Sketches of the interaction of a temperature gradient with a turbulent wake. (a) The upstream conditions and the velocity in the wake. (b) The temperature perturbations in the diffusivity profiles with and without free-stream turbulence.

gradient of the diffusivity [which does not necessarily mean that the magnitude of θ is sensitive to the profile of $K_e(y)$].

The eddy diffusivity $K_e(x, y)$ for the wake of a heated circular cylinder has been found to have the self-preserving form: $K_e = \text{Pr}^{-1} \nu_e [y/(xD)^{1/2}]$, where Pr is the turbulent Prandtl number, D is the diameter of the cylinder, and ν_e is the eddy viscosity function plotted in Hinze (Ref. 3, Figs. 6-7). The value $\text{Pr} = 0.5$ is appropriate to the present flow (Ref. 3, p. 509). The effect of the weak turbulence in the free stream, is modeled by a small diffusivity equal to $\epsilon \text{Pr}^{-1} \nu_e(0)$ outside the wake.

B. Self-preserving solution

We substitute the following self-preserving functions into (1a):

$$K_e = \hat{K}_e h(\eta), \quad \Theta = \frac{dT}{dy} l(x) g(\eta), \quad (2)$$

$$\eta = y/l(x), \quad l(x) = (cxD)^{1/2},$$

where Θ rather than θ has been chosen as the similarity variable for ease of subsequent analysis, and where c and \hat{K}_e are constants which depend on the magnitude of the dimensionless function $h(\eta)$, dT/dy is the uniform upstream temperature gradient, and η is the similarity variable. After rearrangement, (1) becomes

$$\frac{d}{d\eta} \left(h(\eta) \frac{d}{d\eta} g(\eta) \right) + \frac{1}{2} \left(\eta \frac{d}{d\eta} g(\eta) - g(\eta) \right) = 0, \quad (3)$$

provided we choose c such that

$$\hat{K}_e / (cUD) = 1. \quad (4)$$

The boundary conditions on (3) are

$$g(\eta) \rightarrow \eta \text{ as } \eta \rightarrow \infty, \quad g(0) = 0. \quad (3')$$

The first condition states that the uniform temperature gradient is undisturbed far from the wake. The second follows from symmetry [it could be replaced by $g(\eta) \rightarrow -\eta, \eta \rightarrow -\infty$].

A convenient form for the eddy diffusivity, which is also a good fit to the data in Hinze³ (when $\epsilon = 0$), is

$$h(\eta) = 1; \quad 69/65 > |\eta|,$$

$$h(\eta) = 1.69 - 0.65|\eta|; \quad 69/65 \leq |\eta| \leq 2.6 - 1.54\epsilon, \quad (5)$$

$$h(\eta) = \epsilon; \quad 2.6 - 1.54\epsilon < |\eta|.$$

A solution to (3) and (3') after substituting (5) is

$$g(\eta) = C\eta; \quad 0 < \eta < 69/65$$

$$g(\eta) = A(\eta - 1.3) + B \exp(\eta/1.3); \quad 69/65 \leq \eta \leq 2.6 - 1.54\epsilon$$

$$g(\eta) = \eta + D \{ \exp[-\eta^2/(4\epsilon)] - \frac{1}{2} \eta(\pi/\epsilon)^{1/2} \text{erfc}[\eta/(2\epsilon)^{1/2}] \}; \quad 2.6 - 1.54\epsilon < \eta, \quad (6)$$

with $g(-\eta) = -g(\eta)$. The constants A , B , C , and D are obtained by matching $g(\eta)$ at $\eta = 69/65$ and $2.6 - 1.54\epsilon$.

It is found that, if $\epsilon = 0$,

$$A = 0.1064, \quad B = 0.3332, \quad C = 0.6865, \quad D = 0; \quad (6')$$

and if $\epsilon = 0.325$,

$$A = 0.1184, \quad B = 0.3708, \quad C = 0.7637, \quad D = -38.69.$$

For future reference, we introduce the self-preserving temperature perturbation function

$$G(\eta) = g(\eta) - \eta, \quad (7)$$

where

$$\theta = \frac{dT}{dy} l(x) G(\eta),$$

and note that the minimum value of $G(\eta)$ occurs at $\eta = \hat{\eta} = 1.62$ when $\epsilon = 0$ and at $\hat{\eta} = 1.47$ when $\epsilon = 0.325$. The minimum values of G are $G(\hat{\eta}) = \hat{G} = -0.43$ and -0.30 , respectively.

C. Comparison with experiment

Alexopoulos and Keffer expressed their measurements of the mean temperature perturbation in a similarity form as

$$\bar{\theta}_e(x, y) = \theta_0(x) k(\eta_\theta), \quad (8)$$

where $\eta_\theta = y/l_\theta$. $k(\eta_\theta)$ and l_θ are defined by $|k| \leq 1$ and the fact that $k(\eta_\theta)$ reaches its maximum value of 1 at $\eta_\theta = 1$. It was found experimentally that $l_\theta \approx 0.36(xD)^{1/2} = [0.36/(c)^{1/2}] l(x)$ and that $\theta_0(x) \approx -0.4 l_\theta dT/dy$ (the similarity form being attained for $x/D \gtrsim 100$). A value for c may be obtained from (4) and the empirical formula $\hat{K}_e = \text{Pr}^{-1} 0.016UD^{3/2}$. Thus $c = 0.032$ and $l_\theta(x) \approx 2.0l(x)$. Experimental and theoretical values for $\hat{\eta}$ and \hat{G} , denoted by the subscripts (e) and (t), are now compared.

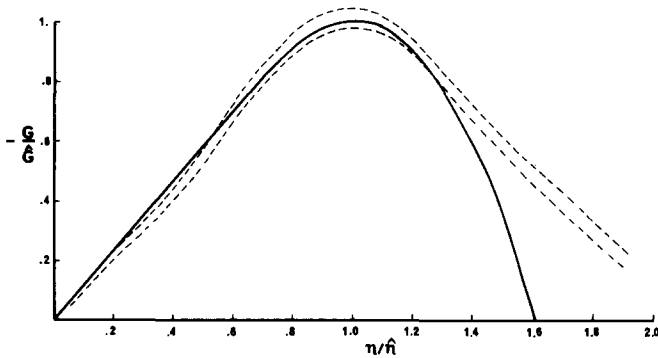


FIG. 2. Calculated and experimental profiles of the ratios of the temperature perturbations to their maximum values: — G/\hat{G} or θ/θ_0 as calculated; ---- Data of Alexopoulos and Keffer. Their data were collected at $x/D = 100, 164, 228$. The spacing of the dashed lines is indicative of the scatter of these data. The abscissa is normalized by the value of η at which G or θ is a maximum.

Since $y = l_0$ when the perturbation is a maximum, $\hat{\eta}_{(e)} = l_0/l = 2$, compared with the theoretical values $\hat{\eta}_{(t)} = 1.62$, and 1.47 without and with free stream turbulence. Similarly, from (8) and (2) [observing that $k(\hat{\eta}) = 1$], the maximum experimental value of G is $\hat{G}_{(e)} = \theta_0/(ldT/dy) = -0.4l_0/l = -0.8$; whereas the theoretical values are $G_{(t)} = -0.43$ and -0.30 . For the ratios $(\hat{G}/\hat{\eta})$ we find $(\hat{G}/\hat{\eta})_{(e)} = -0.4$, whereas $(\hat{G}/\hat{\eta})_{(t)} = -0.26$ and -0.20 , respectively. These ratios are independent of c . In Fig. 2 we have plotted our solution (6) in the form $G(\eta/\hat{\eta})/\hat{G} = k(\eta/\hat{\eta})_{(e)}$, along with Alexopoulos and Keffer's data for $k(\eta/\hat{\eta})_{(e)}$.

It should be remembered that G is the temperature perturbation, and that the similarity form g of the total temperature is $g = G + \eta$. Thus, while Fig. 2 clearly shows that our eddy diffusivity approach is inadequate near the edge of the temperature wake, the total temperature is only 25% in error (without allowing for free stream turbulence).

One correction that needs to be applied to the theoretical results is to allow for the neglect of the mean velocity perturbation (u, v) in the wake. If this correction is $\Delta\theta$, then from (1) $\Delta\theta$ can be calculated by a perturbation analysis from our previous result for Θ ,

$$U \frac{\partial \Delta\theta}{\partial x} - \frac{\partial}{\partial y} \left(K_e \frac{\partial \Delta\theta}{\partial y} \right) = -u \frac{\partial \Theta}{\partial x} - v \frac{\partial \Theta}{\partial y}. \quad (9)$$

An approximate integration of this equation along the line in the wake where $\partial\theta/\partial y = 0$ (i.e., $\eta = \hat{\eta}$) shows that the maximum value of $\Delta\theta$ is

$$\Delta\theta \approx l_0 \frac{dT}{dy} \frac{u}{U} \frac{\hat{G}}{\hat{\eta}}. \quad (10)$$

For this turbulent wake $u/U \approx 1.2/(x/D)^{1/2}$. Thus, the corrected theoretical prediction for the maximum perturbation temperature in the wake is

$$(\hat{G}/\hat{\eta})_{(e)} \approx -0.26[1 + 1.2/(x/D)^{1/2}],$$

as compared with the experimental value of about

$$(\hat{G}/\hat{\eta})_e \approx -0.4[1 + 1.2/(x/D)^{1/2}]$$

deduced from Alexopoulos and Keffer's Fig. 8.

There are obviously significant discrepancies between the results of this theory using an eddy diffusivity model of heat transfer and the experiments of Ref. 4. Could the parameters of the model be changed to improve the agreement? If the maximum value of K_e , \hat{K}_e , is increased, the maximum perturbation G would be increased, but its location would not be. Such an increase in K_e might have been produced by free stream turbulence. Modeling the free stream turbulence by a small additional diffusivity outside the wake increases the discrepancy. Choosing other forms for $K_e(x, y)$, for example, by localizing the eddy diffusivity gradient near the outer edge of the wake is also no better.

The discrepancy is almost certainly caused by the failure of the eddy diffusivity concept itself. We are more inclined to believe the latter because Alexopoulos and Keffer specifically stated that their turbulence was the same as in the other wake experiments where the wake was heated. The criterion for using eddy diffusion is that one considers dispersion times large compared with the Lagrangian time scale (T_L) of the flow; or dispersion over spatial scales greater than the turbulence length scale L_y (Ref. 6). One also needs the turbulence to be nearly Gaussian.⁷ However, since the present flow is continually entraining heated fluid, the time for dispersion of recently entrained fluid is always small compared with the local T_L ; also, the thermal wake thickness is of the same order as L_y . It is particularly near the edge of the wake, where warmer and cooler fluid is being entrained by large eddies⁸ that eddy diffusion fails, a point confirmed by recent experiments by Sreenivasan, Tavoularis, and Corrsin.⁹ {A possible remedy is to consider the flux F , as an integral of the gradient over typical eddy scales, e.g., $F_y(y) = \int_{-\infty}^{\infty} [K(y, y')(\partial\theta/\partial y)(y')(dy/L_y)]$ (Ref. 10).}

In "second-order" modeling of heat fluxes in turbulent flows, the heat flux does not simply depend on local gradients and so it is possible that such methods may be applicable to this problem. But, the aim of this paper is largely to show the advantages and limitations of eddy diffusivity models in two fairly well defined experiments.

The similarity of the self-preserving analysis developed here can be applied to the mixing produced by wakes of three-dimensional obstacles. We estimate that $\theta \approx 0.5l(dT/dy)$, where $l \propto x^{1/3}D^{2/3}$.

III. TEMPERATURE VARIANCE

Our present understanding of temperature fluctuations, even in homogeneous turbulence¹¹ is too poor for one to have any confidence in an analysis of fluctuations within the wake. We therefore restrict our consideration to the "reference" measurements made by Alexopoulos and Keffer of temperature fluctuations produced from a uniform mean temperature gradient by homogeneous grid turbulence.

There are considerable discrepancies among published measurements in this area.¹² Indeed, there are inconsistencies within the data of Ref. 4. However, a significant observation which they make is that fluctua-

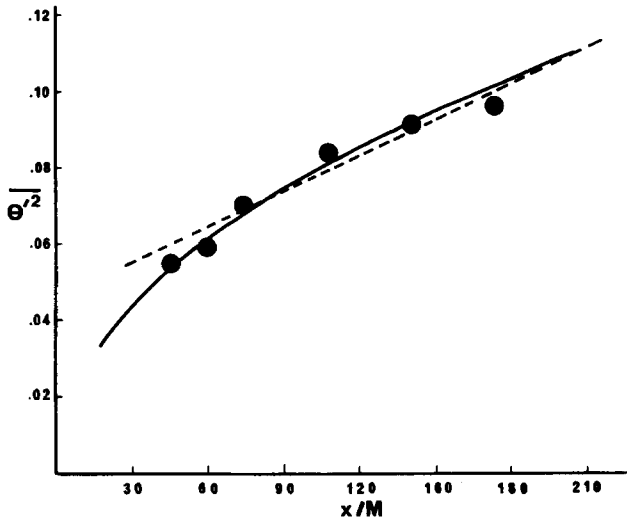


FIG. 3. Experimental values of temperature variance $(\theta')^2$ in $^{\circ}\text{C}^2$ (●), fit with the curves $(\theta')^2 \propto (x-x_0)^2$ - - - - and $(\theta')^2 \propto x^{1/2}$ ———.

tions grow steadily downstream of the grid.

The explanation for this result is that a probe moving downstream from the grid samples fluid particles which have come from ever increasing distances, and hence have an ever increasing range of temperatures. Thus, at points farther and farther downstream, a probe is sampling an increasing range of temperature fluctuations. Quantifying this model using the "one-particle" theory of turbulence diffusion leads to the result, for the temperature variance

$$\overline{(\theta')^2} \propto \left(\frac{dT}{dy}\right)^2 Mx \quad (11)$$

at large x , where M is the mesh size.¹³ Sullivan¹⁴ analyzed and selected some of the measurements in Ref. 4, and from them produced the data for $\overline{\theta'^2}$ vs x shown in Fig. 3. He fitted these data to a straight line; but, he had to assume an effective origin at $x = 100M$. He then suggested that this was confirmation of Corrsin's theory.¹³ However, that theory ignores dissipation of temperature variance. In addition to producing large fluctuations, the mixing of fluid particles with increasingly different temperatures, described here, also increases the smearing of $(\theta')^2$. An analysis⁵ which accounts for this effect, predicts the asymptotic behavior

$$\overline{(\theta')^2} \propto \left(\frac{dT}{dy}\right)^2 M^{3/2} x^{1/2}. \quad (12)$$

In other words, $(\theta')^2$ still grows, but more slowly than in the nondissipative model. The curve $(\theta')^2 \propto x^{1/2}$ also fits the data of Fig. 3. It is inevitable that $(\theta')^2$ scales on $(dT/dy)^2$. (This is something which ought to be tested experimentally.) The constant for Eq. (12) describing the experiments of Ref. 4 is about 0.08.

Of course, the question of whether $(\theta')^2$ increases as x or as $x^{1/2}$ asymptotically is somewhat academic, for the theories predicting these behaviors assume nonde-

caying turbulence. However, there is so little consensus among available data that one cannot even say definitely whether or not, in general, fluctuations will continue to grow after $x/M \approx 50$ (Ref. 12, p. 523).

Clearly, there is much to be explained here and consistent, reliable measurements are required. A real stumbling block in modeling the evolution of temperature variance is the dissipation term. An experiment such as that just discussed can be of real value in understanding this term. (Second- or third-order modeling methods cannot as yet unambiguously predict temperature variance.¹¹)

Returning to the wake flow: On the basis of the reference experiment, one might suggest a similarity form for variance, such as

$$\overline{(\theta')^2} = x^{1/2} D^{3/2} \left(\frac{dT}{dy}\right)^2 \text{ function } (\eta).$$

Although Alexopoulos and Keffer were skeptical about the significance of their results, they did find a similarity form for $(\theta')^2/(\theta')^2_{\text{max}}$ as the result given here suggests. A consequence of similarity is that the "local equilibrium" approach, which equates production of $(\theta')^2$ with the dissipation of $(\theta')^2$, cannot be valid for this flow (convection and diffusion must be as large as production). We have examined similarity models for $(\theta')^2$, but present data are not adequate to justify such models.

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