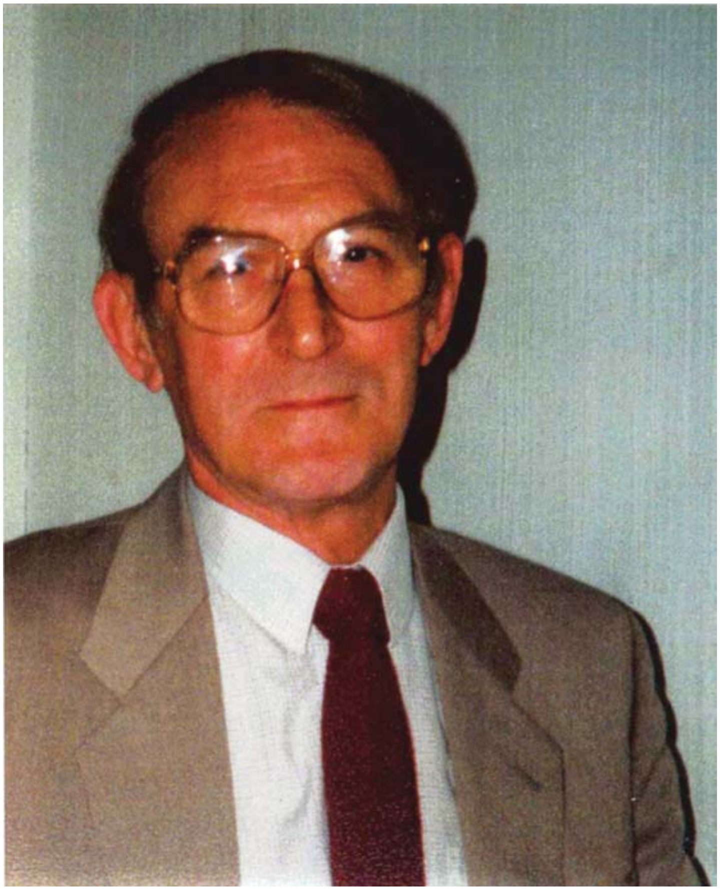

T. Brooke Benjamin

Nonlinear and Wave Theory Contributions of T. Brooke Benjamin (1929–1995)*

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bubbles, waves, instabilities, nonlinear analysis, swirling flows

Abstract

Brooke Benjamin's original theories of fluid mechanical phenomena changed our basic understanding of cavitation bubbles, surface and internal waves, gravity currents, instabilities of shear flow over flexible surfaces, and swirling flows. For some types of finite-amplitude wave phenomena, he generated integral constraints and derived new partial differential equations; by establishing their general properties he showed how they have wide application. He developed a complementary approach based on functional analysis that was quite new to fluid mechanics. He demonstrated methods for deriving, without detailed calculation, the essential features of nonlinear and indeterminate flow problems that are otherwise intractable.

OVERVIEW AND PERSONAL LIFE

T. Brooke Benjamin was a brilliant researcher whose original and elegant use of applied mathematics and extraordinary physical insights changed the way we now study and think about fluid mechanics. He did this largely through deep analysis of particular problems such as the similarities in movements of sea breezes, avalanches, and long air pockets in tubes; why thin vortices on aircraft wings can behave like shock waves and suddenly become thicker and change the whole flow; how tiny bubbles collapse and hence cause the damaging erosion found on propellers; how the flexible skins of dolphins and, nowadays, "smart" submarines can damp out fluctuations in the turbulent flow over their skins to increase their speed and reduce the noise heard by nearby prey or predators. This is perhaps a surprising list of accomplishments for the Sedleian Professor at Oxford University, but not when you consider Brooke Benjamin's remarkable life and career. When his life was tragically cut short with cancer in 1995 he was studying the unpredictability and hysteresis of flow patterns in a lubrication bearing around a rotating shaft, and in doing so still making important new discoveries about the deepest mysteries of fluid mechanics.

Thomas Brooke Benjamin was born on April 15, 1929 in Wallasey Cheshire, the eldest of three children. He developed musically, saving up to buy a secondhand violin and writing a number of musical compositions, notably a piano quintet performed at a concert in Wallasey when he was 16 and a string quartet in 1948. He graduated from Liverpool University with a Bachelors in Engineering in 1950 (with first-class honors), where he was also active in the Liverpool University Music Society and conducted numerous performances in Liverpool and elsewhere, frequently of his own compositions. By this time he had developed a strong social and political awareness and spoke often in public as a pacifist and independent, a useful experience for his later involvements in public controversy as a professor at Essex University and on national committees.

Faced with a choice between music and science as a profession, he chose science. In 1951 he was granted a Rotary Foundation Fellowship to study electronic thermionic valves (the development of the transistor being "round the corner") at Yale University, where he gained a Masters in Engineering in 1952.

In October 1952 he was accepted as a research student at King's College, Cambridge, to continue his electronics research in the engineering department. But, as the story goes, he popped his head round the door of the hydraulics laboratory, where he saw experiments on swirling flow in tubes, the unusual square-shaped G.I. Taylor wave tank, and A.M. Binnie clambering around the huge pipes and tanks. Benjamin was fascinated and stayed to work with Binnie on a new subject for the laboratory, the formation of the tiny cavitation bubbles that form in high-speed water flow and their sudden collapse. They used electronics to trigger the bubbles and to take remarkable high-speed photographs of individual bubbles.

A characteristic feature of the elegant theories Benjamin built on these experiments, and on his other studies of avalanche-like gravity currents, steep subsurface internal waves in the ocean, and swirling flows, was the use of appropriate averages (in fact, integrals of mathematical functions of the velocity and pressure over spatial volumes) to explain the underlying principles of these and many other hydrodynamic problems. He was inspired by the nineteenth century pioneers of this approach, Sir W.R. Hamilton and Lord Kelvin. The latter remarked in 1879 that "his [Hamilton's] method of varying action undoubtably must become a most valuable aid in future generalisations."

Benjamin's other great general idea was that the hydraulic jump (that one sees behind a big stone in a stream or in the kitchen on the rim of a plate placed under a tap) is in fact an ubiquitous model for transitions in flows between states where waves can and cannot travel upstream, such as in rotating, stratified, and bubbly flows. From these platforms his researches dug deeper into questions about flows that were ever more general and abstract, beginning with engineering questions about the total energy or impulse[1] of the flow, and later moving to questions about whether particular types of flow can exist or not, or whether they can jump from one state to another. One might also trace the movement of his mathematics toward more abstractness in relation to the growing capacity of electronic computers to calculate flows with ever greater detail. Despite some skepticism at Cambridge and elsewhere, Benjamin pioneered the way for applied mathematicians to ask and answer more general questions, particularly as computers began to replace their earlier role of making specific calculations. Researchers would still do well to read his papers, which provide an unrivaled introduction to the underlying ideas and questions of many fundamental aspects of fluid mechanics.

On the basis of his thesis Benjamin was elected to a fellowship at King's College in 1955. He had already published the first of his papers, on the emerging topic of nonlinear effects in currents and on steep water waves, coauthored with James Lighthill (with whom he also played duets on his grand piano at Kings). This led to his appointment in 1958 to the specially created post of Assistant Director of Research, held jointly between the engineering department and the embryonic department of applied mathematics and theoretical physics. He helped set up a new fluid mechanics laboratory in the old Press building in 1964 and was appointed Reader in 1967.

During this period he widened his interests in hydrodynamical problems as an active member of the illustrious group of researchers drawn together by G.K. Batchelor and G.I. Taylor and renowned for the ferocious arguments of their Friday afternoon seminars.

Benjamin's teaching of undergraduates at Cambridge was minimal, but no one who attended his lectures can forget his deliberate entry into the room, the precise wiping of the board, the ceremonial breaking of chalk, the elegant thin writing with serifs on the letters, and the quiet, economical explanations as he led his class through the analysis of water waves.

In 1970 he left for a chair in mathematics at Essex University, where he set up the Fluid Mechanics Research Institute funded by the Science Research Council. His particular aim was to show how more abstract mathematical analysis could, as the

[1]An early literary description of impulsion associated with vortices is found in Dickens' 1865 novel *Our Mutual Friend*.

pioneering Russian school had already begun to demonstrate, contribute to a deeper understanding of different types of flow (e.g., solitary waves, instabilities, rotating flows). One example was the well-defined, but difficult to understand, flow between rotating concentric cylinders, in which the flow between them sets up slow eddying motions with several different possible flow patterns. Although the Institute was in the mathematics department, he continued the Cambridge tradition of having an experimental laboratory within the Institute, not just to check theory, but to explore basic ideas and to stimulate theory. He did a remarkable job in bringing together different sides of the mathematical community.

During his time at Essex University he also took an active interest in the turbulent campus politics of the 1970s and in conducting the university choir.

Upon his election to the Sedleian Chair of Natural Philosophy at Oxford University in 1979, and a fellowship at The Queen's College, Benjamin continued the approach he had developed at Essex in the Mathematical Institute. He and his close collaborator Tom Mullin showed how it could be applied to interpreting sensitive experiments where many different types of motion are observed. He returned to a number of the hydrodynamic problems he had studied earlier in his career, such as vortex breakdown and bubble dynamics, and in so doing approached the problems with more general and powerful mathematical apparatus. He also participated in some of the famous Oxford seminars on mathematics applied to industrial problems, led by the late Alan Tayler.

He held a number of visiting appointments abroad, mostly in the United States, where in the 1960s he wrote some of his seminal papers while visiting the Institute of Geophysics and Planetary Physics, at the University of California, San Diego and the University of Michigan. He spent sabbaticals at the Universities of Wisconsin and Houston in the 1980s. But he returned most often to Pennsylvania State University, where he was an adjunct professor and where a symposium was held in his honor to celebrate his 60th birthday. He was an enthusiastic follower of American football and enjoyed watching Penn State's successful football team.

Benjamin was a great supporter of scientific and academic communities throughout Europe. He visited French institutes and was elected as a foreign member of the French Academy of Sciences in 1992. He actively supported the U.K. National Conference of University Professors and was elected as their founder President. He enjoyed public debate and contributed controversial articles in the national educational press. He particularly urged the British government to recognize the value of its universities. He admired many aspects of the approach to higher education in the United States, from its grading schemes to how the wider community became more involved in university life, especially the fortunes of their football teams! Elected a Fellow of the Royal Society in 1966, he was further honored by being elected a member of Council twice and being invited to give the Bakerian Lecture in 1992 on the "mystery of vortex breakdown." He received honorary doctorates from the Universities of Bath, Brunel, and Liverpool in 1966. In 1969 he received the L.F. Moody Award from the American Society of Civil Engineering and the William Hopkins prize of the Cambridge Philosophical Society.

Brooke Benjamin had a most attractive aura—wise, sensitive, and with a touch of mystery—in fact, a bit of a Cheshire cat. He was about 6 feet tall, moving slowly, smiling slightly, often with a pipe in his mouth, and speaking so that every word counted. He and his first wife Helen Ginsburg, who met at Berkeley in 1952, and married in 1956 (they divorced in 1974), entertained many students in their Grantchester home near the river Cam. They had three children, Lesley, Joanna, and Peter. He and his second wife Natalia Court (who taught him French) married in 1978. Their daughter Victoria was born in 1982. They were a joyous couple, welcoming many friends from all over the world to their friendly home in North Oxford. Most people never knew that he was a diabetic. In his final illness he remained cheerful, optimistic, and forever curious, still talking about bubbles.

DYNAMICS OF BUBBLES AND CAVITATION

Benjamin's first area of research, and one that he kept returning to later, was the behavior of bubbles in liquids. For his Ph.D. Benjamin (1955) studied the collapse of individual vapor cavities or cavitation bubbles, which form in high-speed liquids when the local pressure drops below the vapor pressure at that temperature. When the pressure rises these bubbles collapse very rapidly, emitting noise. It was surmised that this was the mechanism whereby cavitating bubbles cause great damage to nearby solid surfaces, such as ships' propellers or concrete harbor caissons. Before Benjamin's research it was neither clear how this damage occurred nor how to calculate the phenomena. Previous studies by Rayleigh in 1917 and the U.S. Office of Naval Research in the 1940s (e.g., Plesset & Mitchell 1956) had shown how collapsing bubbles cause very high oscillating pressures, which produce the noise. But in their analysis the bubbles remain perfectly spherical at all times. Benjamin's most original contribution was to show that as bubbles collapse and oscillate their shape may undergo enormous distortions. They cease to be spherical, with one side of the bubble indented, thus leading to a rapidly moving jet of vapor penetrating the other side and exiting from the bubble with a sharp point. It is the rapidly moving liquid surrounding the jet that causes the damage to any nearby solid surface. Benjamin constructed a simple experiment to create a vapor bubble of about 2 cm, which was then disturbed. It was recorded with very high-speed photography (over a period of less than 1 millisecond). In this, as in his other experiments, the advanced design reflected Benjamin's early training as an electrical engineer. The photographs of the jet emerging from the bubble were the first to show this phenomenon [it had been suggested previously as a possibility by Kornfield & Suvorov (1944)].

His Ph.D. experiments were extended later (Benjamin & Ellis 1966) to cavitation bubbles near a plane boundary. In that case the distortion of the bubble led to formation of a vortex ring, topologically as drastic a change as the previous case of a penetrating jet, although the peak velocities are less in this case and therefore less damaging. Even if that were possible, these remarkable phenomena could not be understood simply by calculating the details of the bubble deformation. As a qualitative explanation for the whole flow phenomenon, Benjamin took Kelvin's concept

[Thomson & Tait (1879), although attributed by Kelvin to Kirchoff (Kirchoff 1869)] of the impulse (or overall momentum) and applied it to the unsteady motion of the fluid surrounding the collapsing and moving bubble, together with the force applied on the fluid by the boundaries. This led to a constraint and specific predictions, for example that, if a moving bubble collapses, it cannot do so in the same way as for a stationary bubble, i.e., shrinking to a point. Because the momentum of the liquid is preserved, the only possibility is that the bubble's shape and topology change so that it becomes an annular bubble (like a ring doughnut), with the fluid motion becoming more like that of a moving vortex ring—one of Kelvin's "atoms" of fluid flow. This was one of the first occasions that anyone had combined, with some mathematical rigor, both topological and dynamical analysis to fluid flow problems, which Benjamin extended further by considering topological changes to the surfaces bounding the regions of fluid motion. Others have followed him (e.g., Tobak & Peake 1982).

The high-speed photography of the two types of deformed bubble predicted by the new theory were also high points in bubble dynamics for many years (e.g., Blake & Gibson 1987). Benjamin (1958) recognized the role of computers as a vital exploratory tool for advancing fluid mechanics early in his research on bubbles in his collaboration with C. Hunter, who was working in the Cambridge Computer Laboratory. Benjamin's paper on the theoretical pressure waves from collapsing cavities was mainly a mathematical analysis of the distortion of spherical bubbles when pressure waves grew so large that shock waves began to form. He had to extend the linear theory of Rayleigh and others to include the nonlinear effects that cause shocks. This was an early application in a nonaeronautical field of the recent methods introduced by Lighthill of nonlinear stretching of coordinates to convert an intractable nonlinear partial differential equation into a linear one. But in the ultimate phase of the bubble's collapse to its central point, his mathematical analysis failed and a computer solution was necessary. This indicated that the radial velocity of the collapsing cavity becomes so large that the continuum approximation breaks down, a phenomenon that most recently explained the phenomenon of sonoluminescence.

The major theme of his later papers on bubble dynamics was extending integral methods to calculate at least in principle the velocity fields and bubble motion. Benjamin applied Hamiltonian theory to bubbles, although, following Zakharov (1968), others had already shown the power of these methods to other problems in hydrodynamics such as analyzing water waves and the stability of flows of liquids with free surfaces. As with his other integral analyses, he derived the key feature of the different forms of bubble motion all within a single framework and in terms of a few parameters (the energy E, impulse I, and impulsive couple L). Thereby Benjamin related the complex variety of oscillations and spiraling motions of rising bubbles and their effects on the surrounding flow (e.g., long trailing vortices). For recent computations see Mougin & Magnaudet (2002). As Benjamin noted, some specific results had been or could be derived by the usual deductive methods starting from particular assumptions and boundary conditions, but then one loses the overall perspective of the relation between the various phenomena. A striking feature of these unsteady bubble oscillations also described in the general analysis is that they can cause the translation of bubbles in chaotic patterns, a phenomenon of "dancing bubbles." The original hypothesis of Saffman (1967) was confirmed by Benjamin & Ellis (1990) and related precisely to the impulse and the mode of oscillation of the bubble. Laboratory experiment on 1-mm-diameter bubbles subjected to 5-kHz oscillations nicely confirmed their self-propulsion at different angles depending on the mode being excited.

To clarify the basic concept in these integral formulations of inviscid flow problems involving moving rigid or deformable bodies, Benjamin (1986b) showed how the impulse integral I is a delicate mathematical quantity that depends sensitively on how the three-dimensional integrals are evaluated (as they are not absolutely convergent). By considering the volume of fluid transported by a moving body (the drift volume of Darwin 1953), the effects on I of free or rigid boundaries infinitely far from the body on the impulse were determined. This note stimulated later studies (e.g., Eames et al. 1994).

NONLINEAR SURFACE WAVES

Even before his Ph.D. thesis was complete Benjamin progressed from the study of the motion of the curved finite area "free" surface between gas and a liquid, the bubble problem, to that of the larger-scale infinite area interfaces, the water wave problem, where the dominant restoring force causing oscillations is not surface tension but gravity. In making this transition he first developed his ideas of characterizing the main features of flows in terms of their integral properties, and showed how, in many situations where waves exist, the local flow structure can best be understood in terms of the generation and propagation of waves passing into and out of the local region.

Rayleigh (1910) and Taylor (1910) both analyzed the general tendencies of steepening of sound waves that may arise whenever higher levels of pressure are propagated at a greater wave speed than lower levels. They showed that, at the moment when this tendency is causing wave steepness to grow without limit, dissipation effects become large and act to bring wave steepening to a halt. A sound wave of finite amplitude then develops a microscopically thin shock wave (almost, but not quite, a discontinuity); furthermore, the dissipative effects account fully for the known amount of entropy increase at a shock wave. Before 1910 this was calculated as a discontinuity.

After the enigma was resolved for sound waves, the next obvious challenge was to do the same for water waves. Rayleigh (1914) had shown that there is a general tendency for surface waves on shallow water to steepen and calculated the amount of mechanical energy loss that must happen if, once again, the steepening proceeds until a discontinuity (bore or hydraulic jump) is formed. But the question remains as to whether there is a simple analogy with sound waves; in other words, whether dissipative effects may succeed in opposing the tendency of waves to steepen and jumps to be formed.

The complicated answer (see below) to this question came simultaneously from Lighthill at Manchester University, and Benjamin, who had just started postdoctoral research in the Cambridge University Engineering Department (Benjamin & Lighthill 1954).

Linear surface waves on shallow water differ from sound waves in being slightly dispersive, so that not only does the wave speed fall slightly as the wavelength is reduced, but also the group velocity (or velocity of energy propagation) is a little less than the wave speed (or speed of propagation of the crests). Thus, if a bore or hydraulic jump travels along with a train of periodic waves behind it (the crests traveling at a wave speed identical with the speed of travel of the bore), the energy in those waves travels more slowly. Therefore, relative to the bore itself, there is a backward transport of mechanical energy, a leaking away of energy from the region of the bore. In simple physical terms, this mechanism (which cannot arise in sound waves) may be responsible for much of the needful energy loss calculated by Rayleigh. A key feature of this analysis was a conjecture about the interdependent possible ranges of Q, R, and S, the volume flux per unit space, the total head, and the flow force (per unit span divided by the fluid density). This "pivotal conjecture" was later proved in one of Benjamin's (1995) last papers, using the rigorous bounds established by Keady & Norbury (1975).

Benjamin & Lighthill showed that whatever the amplitude of the hydraulic jump, the nature of the periodic waves behind it, within which the energy leakback can balance (or nearly balance) this needful energy loss, is such that linear theory cannot correctly be applied to calculate their properties. They are subject instead to the equation pioneered by Korteweg & de Vries (1895), which had been almost forgotten at the time of the 1954 Benjamin & Lighthill paper (even though it was to develop into one of the most celebrated equations of the twentieth century's later decades). That equation's important solution in the bore context is not the now famous soliton, but rather the cnoidal wave, characterized by the fact that the classical sinusoidal shape is replaced by a shape specified by the Jacobian elliptic function *cn*.

Another path-breaking property that both authors independently observed was that, in the frame of reference with crests stationary, a cnoidal wave train possesses extremely simple values for three well-defined quantities: the rates of flow of mass, momentum, and energy per unit length of crest. Analysis of a bore that carries behind it a cnoidal wave train (with crests stationary relative to the bore) may then be carried out very simply: The rates of flow of mass and momentum in the oncoming uniform stream are equated to those in the downstream cnoidal wave train, whose rate of energy flow can be given any value less than or equal to that in the oncoming stream (according to whether or not any dissipation occurs).

The analysis explains certain surprising observations about undular bores (those relatively weaker bores with an essentially smooth water surface that carry a train of waves behind them). These observations show that, at any given bore amplitude, the range of wave amplitudes can be surprisingly large, varying from 0.3 to 0.9 times the bore amplitude. (Here, bore amplitude means jump in mean water level, and wave amplitude means half the vertical distance between crests and troughs.) These amplitudes correspond to rates of energy dissipation under different experimental conditions, ranging from 75% to 5% of the required energy loss (so that the remainder of that loss, which takes the form of energy leakback in the cnoidal wave train, assumes proportions from 30% to 95%).

The general idea of characterizing any steady motion in terms of rates of flow of mass, momentum, and energy within it was to prove particularly influential, and, of course, to be used extensively by Benjamin in later papers (which he reviewed in Benjamin 1984).

Benjamin began his research on the unsteady behavior of surface waves in a joint paper (Benjamin & Ursell 1954) on highly controlled waves on the surface of a body of liquid held in a rigid oscillating container with a horizontal base and vertical walls. In his classical experiment, Faraday (1831) noticed that the frequency of the liquid was only half that of the vessel. Later data showed sychronicity. This Benjamin-Ursell joint paper completed Rayleigh's (1883) practical explanation of these divergent results, using the later theory of Mathieu functions for the forced vibration of oscillatory systems (originally pendulums). Essentially, the frequency depended on the type of mode. Some careful experiments by Benjamin in the Cambridge Engineering Department confirmed the theory for the Faraday half-frequency mode. In this and his other papers Benjamin carefully justified the use of ideal theory to describe real fluid experiments by detailed discussion of the errors due to viscous processes.

In his most celebrated work on waves Benjamin shared the experience of many great scientists in making a discovery from an experiment that went wrong. Although the basis of Stokes' (1845) theoretical analysis of small water waves of permanent form had been confirmed by the existence theorem of Levi-Civita (1925) and by later authors (reviewed by Benjamin 1967b), by the 1960s no one had demonstrated the stability of these waves, or even had "suspected . . . that [their] equilibrium . . . is in fact unstable." This was not only a question of fundamental interest but also of practical importance for the design and operation of wave tanks (e.g., used for ship design), where it was assumed that the frequency of waves could always be controlled as accurately as required by suitable adjustments of the wave maker. By the 1960s nonlinear interaction between oscillations of different frequencies and wavelengths was studied in many different systems such as electronics, solid-state physics, plasmas, turbulence, and water waves. In the latter types of continuous systems most theories showed that finite-amplitude pure oscillations would break down into higher and lower harmonics, as Phillips (1960), Longuet-Higgins & Phillips (1962), Benney (1962), and Whitham (1966) had shown for particular cases. The essential questions were whether there might be some special property of the interaction of a train of sinusoidal water waves that could suppress the growth of such harmonics or whether, if the harmonics did grow, there could be some resonant interactions between a few harmonics so that the growth was exponential.

In 1963 Benjamin's Canadian research student Jim Feir was measuring in detail the waves along a new 10-m tank in the Engineering Department in Cambridge. He and Benjamin observed how waves generated at the wave maker started as a regular train with constant frequency and wavelength, but then about 5 m down the tank began to form into groups of waves with varying frequencies and wavelengths. (I was a research student in an adjoining laboratory, watching their surprising phenomena, and shared in the excitement as the story unfolded. But this account of the events is largely due to Feir.)

Both of them believed that these first observations might be associated with serious imperfections in the construction and operation of the wave maker because by varying its frequency the curious evolution of the wave patterns could be generated deliberately. Benjamin worked out a simple control system to eliminate any measurable drift in frequency. Still, the wave breakdown observed earlier persisted, and Feir was able to measure wave profiles along the tank for various values of wave slope. These showed an exponentially growing amplitude modulation along the tank at one fifth the wave maker frequency, and in particular suggested that modulation growth ceased at certain cut-off values of wave slope.

Then Feir moved to the much larger wave tank equipped with a programmable wave maker in the National Physical Laboratory (NPL) at Feltham (near London). Several modulating frequencies and amplitudes were imposed on the wave maker. The results of these experiments confirmed what had been observed at Cambridge. The synthesis of a weak-amplitude modulation into a dominant carrier and two small sidebands led Benjamin to the idea of instability arising from wave propagation through a periodic medium. He derived a linear analysis of a deep-water Stokes wave perturbed by two sidebands, leading to predictions of sideband growth rate as the wave slope increased, provided it was greater than a critical value, defined by the ratio of wave height h to wavelength λ equal to $1.363/2\pi$. Both aspects of these results were in good agreement with the experimental measurements.

The theoretical analysis was published by Benjamin & Feir (1967) as part one of a two-part paper. Feir's experimental results, which were meant to be published in part two, were only briefly summarized in one figure of the paper (Benjamin 1967b). Details can be seen in Feir's (1967) Cambridge Ph.D. thesis. Doubt was expressed at the time about the originality of the Benjamin-Feir instability analysis (which Benjamin referred to as the "wave train breaking up into groups"). There had been earlier publications on resonant wave-wave interactions by Phillips (1960), on the general nonlinear analysis of interactions between waves of differing frequency and wavelength by Hasselman (1962), and on other specific wave-wave interactions. But, as Hasselman recognized, he had not applied his analysis to study the basic questions about the stability of Stokes waves. So it is now generally accepted that Hasselman is credited with the first general nonlinear wave analysis and Benjamin with the stability analysis. Also, the nonlinear transition in the behavior of waves at the critical slope had been pointed out earlier by Whitham (1966).

The two complementary discoveries of water waves in the nineteenth century were those of infinitesimal waves and finite-amplitude solitary waves. A group (or packet) of the former waves disperses as it propagates (as one sees when a stone is dropped into a pond), but when the latter larger type of wave is formed, it can propagate as a single wave and without change, even in the slightly perturbed flows along canals where it was first described scientifically by Scott Russell in 1845. The first mathematical analyses of solitary waves were those of Boussinesq (1871) and Korteweg & de Vries (KdV) (1895). Whereas small-amplitude waves were well described by Stokes' (1845) theory (Craik 2005) (although their stability and uniqueness were unknown), mathematical and physical questions had been raised about whether the KdV model equation was the most general and correct description for long waves, including solitary waves.

Nevertheless, many of its mathematical properties had been established, partly because of its relevance to waves in plasma physics. The stability of solitary waves and their sensitivity to variations of the water current on which they were propagating was also an important mathematical question, although it was known experimentally that these solitary waves would not destroy small waves passing through them (Miles 1980), and could, as mentioned before, travel along streams with nonuniform current.

Benjamin's (1962a) attack on solitary waves began with the latter problem, when he used a formal expansion method to show that the variations in the vorticity of the stream on which a long solitary wave was propagating slightly increased its speed. This paper was a powerful generalization of the earlier irrotational calculations of Boussinesq (1871) and Rayleigh (1876). The results could also be applied to a train of waves when each one was similar to a solitary wave (Benjamin & Lighthill 1954).

In 1970 Benjamin moved to the newly formed Fluid Mechanics Research Institute in the Mathematics Department at Essex University, where he brought together a number of distinguished colleagues and research students from the United Kingdom, Australia, and United States, including R.W. Smith, J.L. Bona, J.J. Mahony, W.G. Pritchard, and J. Toland. The research focus of the Institute was an extended examination of the mathematical theory of waves, especially long waves, on a water surface. They were particularly interested in the KdV equation, which, although it is a rigorously correct representation of infinitesimal waves, had heuristically been widely used as an approximation to finite-amplitude long waves (propagating in one direction) both in wave trains and as solitary waves. As a first step Benjamin & Bona (1972) studied the stability of the KdV description of the latter type of waves. They powerfully used the integral or functional approach, drawing on earlier, but largely forgotten, results of Boussinesq and also on Lyapunov's general concepts of stability of systems (asking the question of whether integral properties of the perturbed wave differ significantly from those of the undisturbed wave over time) (as in Benjamin & Lighthill 1954). Benjamin was one of the pioneers of the application of this concept to fluid mechanics, which was just beginning at this time. Incidentally, Benjamin & Mahony (1971) showed how some of the key integral invariant properties of water waves are related to each other, demonstrating, for example, that the centroid of the wave disturbance travels with a constant speed in fluid of constant depth—a useful new result. For the stability analysis, where an inequality of one of the integrals had to be proved, Benjamin used classical spectral theory (i.e., by considering certain eigenvalue problems related to the form of the integrand). He concluded that he had demonstrated that the solitary wave solution of the KdV equation is stable for practical purposes, i.e., any perturbed solution remains very close to the form of the original wave. However, mathematical questions remained that possibly the perturbation would affect the wave at large times.

The next step by Benjamin et al. (1972) was to see whether, following Whitham (1967) and Peregrine (1966), the KdV equation is mathematically and physically the most appropriate model equation for finite-amplitude long waves. A major mathematical drawback of the KdV equation, which can be written in general (nondimensional) form as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

is that it was not possible to prove that the solution actually exists (given the initial perturbation of the wave height over a limited extent on an infinitely wide water surface). Although physicists and engineers are seldom concerned about existence theorems of their model equation, they do worry about a model when mathematicians prove a nonexistence theorem. These practical as well as mathematical reasons drove the Essex team to propose an alternative model, especially one that was appropriate for long finite-amplitude water waves, namely the Benjamin-Bona-Mahony (BBM) equation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2 \partial t} = 0,$$

although it was first introduced, in a slightly different form, by Peregrine (1966). They showed that an existence theorem could be proven for this equation and that it is unique. This confirms, as is observed in experiment, that finite-amplitude waves do not suddenly change into other forms. The result is also consistent with Benjamin's (1972a) conclusion that these solutions are stable.

Despite the successes with the analyses of small-amplitude solitary waves, mathematical questions remained about whether isolated finite-amplitude solitary waves could really propagate unchanged and be stable. Classical analytical methods for periodic waves cannot be applied in these cases. Using topological concepts, "in particular the fixed point index for compact mappings and . . . a version of positive-operator theory referred to in Frechet Spaces," new techniques for tackling these problems were later developed by Benjamin et al. (1990). Benjamin (1974, 1979) applied these to finite-amplitude waves in stably stratified flow. Later research showed that at high amplitudes, when neither of these equations are valid, real solitary waves are unstable (Tanaka et al. 1987).

Benjamin et al. (1987) developed the theory of water waves moving over sand bars and, thence, how changes in beach profiles are influenced by the varying effects of wave reflection and of turbulence caused by breaking waves. At Oxford he further widened the range of mathematical methods for calculating the nonlinear behavior of periodic and solitary water waves. He gave his own particular slant to the use of Hamiltonian methods (i.e., minimizing certain integrals) that the Russian school of applied mathematicians (notably Zakharov 1968) had pioneered in the 1960s. These methods are attractive because of their simplicity in satisfying the nonlinear boundary conditions at the free surface.

FINITE-AMPLITUDE INTERNAL WAVES AND STRATIFIED FLOWS

From the early 1960s Benjamin began to apply the analytical methods of wave theory to unsteady flows with stable density gradients. Benjamin focused on persistent and finite-amplitude long waves, with or without accompanying currents. These phenomena are common occurrences in aquatic and atmospheric environments, where they are often visible as fronts between water masses and cloud formations and are experienced as clear-air turbulence by aircraft. Long waves with sharp fronts known as gravity currents can effectively control certain engineering flows of gases and liquids.

Benjamin's aim was to introduce a new framework for these flows, especially the nonlinear aspects. He built on previous studies by Taylor (1931) on infinitesimal internal waves and on large bubbles, by von Karman (1940) on the general form of the fronts of gravity currents, and by Keulegan (1957) on internal wave experiments. Miles & Howard (1964) had already shown how integral methods could lead to predictions of stability to small disturbances of general kinds of stably stratified shear. Also, the nonlinear equation for finite-amplitude waves (with uniform stratification) had been derived by Long (1953) following Dubreil–Jacotin's study (1937), and Yih (1965) had noted earlier the mathematical similarity between swirling and stably stratified flows.

Benjamin (1966) first tackled the general problem of long two-dimensional waves of unchanging form traveling in a stably stratified incompressible fluid on a unidirectional steady stream with varying velocity. He explained why solitary waves could, as in free-surface flows, be expected in these internal flows because of the contradictory tendency of large-amplitude waves to steepen (on a shallow stream) since long waves travel fastest and small wave packets to disperse. However, he warned about a further factor in these flows (which still tends to be overlooked): the non-Boussinesq effect on the inertial terms of the variations of the density in the flow. This changes the dominant effects of forces in many flows of geophysical and engineering interest. Using the general approach of Benjamin (1962a), the integral here being the flow force S, he solved an integral equation by successive approximations to derive explicit formulae and a general interpretation for the wave properties. This approach enabled him to suppose that, when the upper boundary is fixed or free and in a horizontal flow of a fluid with two layers of unequal density, traveling solitary waves could form. But if they were large enough there could be a dissipative transition between a faster flow (where the velocity was greater than the speed of smaller waves on the interface) to a slower flow (where the velocity was less). This would be analogous to the hydraulic jump (or wavy transition) that occurs in a free-surface flow (Benjamin & Lighthill 1954). At that time this transition had not been observed experimentally, but it now has been, although the results cannot be explained straightforwardly in terms of Benjamin's conjugate states (also see Section 6) (e.g., Klemp et al. 1997, Wood & Simpson 1984).

Benjamin (1967c) continued his analysis when visiting the Institute of Geophysics and Planetary Physics. In thin regions with intense stable density gradients, such as the thermocline below the mixed layer of the ocean, waves exhibit the classical linear properties that long waves travel with a speed that is approximately independent of wavelength. The wavelengths must be long compared to the thicknesses of these regions. Benjamin's nonlinear analyses clarified why finite-amplitude solitary waves traveling along these layers, for example generated by the sudden movement of an obstacle or an eddy impacting on the inversion layer, would be downward (toward the dense layers if there is a free surface above the less dense layer), whereas it is upward on a free surface. For these solitary waves he introduced a new weakly nonlinear integral-implicit equation, $u_t + u_x + cuu_x - \alpha \Lambda u_x = 0$, in which $\alpha, c > 0$. Λ is a linear pseudodifferential operator, for a suitably normalized vertical displacement of the fluid. This is now known as the Benjamin-Ono equation because Ono (1975) also discovered it in Japan. As before, general results for solitary waves and a train of cnoidal waves (c.f., Benjamin & Lighthill 1954) were obtained for arbitrary density and velocity profiles. Similar results with excellent experiments were published simultaneously by Davis & Acrivos (1967). Benjamin (1966, 1967c) was one of the first to point out in a mathematical/physical analysis, without detailed computation, that in these layered stratified flows the wave drag of obstacles, such as mountains, is related to the train of finite-amplitude waves in their lee. Only in their downstream development is the dissipation of energy in these waves a critical factor.

Benjamin (1968) established a general theory for gravity currents. These are a special kind of finite-amplitude long waves moving along free surfaces or between layers of stratified fluids. Downstream of their frontface the currents move with a constant speed; they may or may not have a distinct frontal structure at their downstream end. The main quantitative question is in determining the approximate speeds of the fronts relative to the surrounding fluid, in different types of stratified flow and with different boundary conditions. The buoyancy or gravity forces within the wave or current drive them against the inertia of the fluid that is pushed aside by the current and by the shearing forces caused by viscosity and turbulent stresses.

Benjamin's analysis provides the first general explanation for why these currents have similar properties in varied flow situations. His simple analysis (say, of a dense layer moving on a rigid surface) based on examining the integral properties of the flow momentum and energy flux showed that, as the velocity of the exterior fluid passing over the top or "head" of the density current accelerates, it tends to travel faster (by about 40%) than long waves on the density interface, and then it decelerates downstream of the head.

In some flows, such as long air bubbles moving up a tube filled with water so that water pours out of the end of the tube, the flow is smooth everywhere and the depth of the bubble or gravity current is a fixed proportion (about half) of the diameter of the tube or channel. This prediction and that for the speed of the current agreed well with the laboratory experiments of Zukoski (1966). However, where long density currents occur in the ocean or atmosphere there may be no upper boundary. Benjamin showed that in such cases the speed of the flow far downstream of the front of the current had to reduce to that of a subcritical flow. Such a process necessitates the large energy loss through turbulent dissipation of breaking waves traveling downstream. This was a critical omission, as he pointed out in von Karman's (1940) inviscid analysis. He robustly concluded that the head wave commonly observed at the front of gravity currents is essential, not merely incidental, to the dissipative process. Every aspect of the analysis and physical ideas of this paper continues to inspire researchers, even when there are points with which they do not agree. His own department at Cambridge University and many others followed up on Benjamin's and J.S. Turner's discoveries about gravity currents with 30 years of vigorous research on many new aspects of these flows (see Simpson 1997).

Disturbances in stratified, two–layer, and rotating flows not only propagate downstream with the current, as one sees in the wakes of mountains and ships, but they also propagate upstream. Although the latter effect is well documented in laboratory studies, it is less often visible in nature, e.g., the disturbed water surface upwind of a bridge pier in a slowly moving river. Perhaps because this effect is often too small to observe, it used to be neglected in relevant flow calculations. But as Benjamin (1970) pointed out, these upstream perturbations have a controlling influence on many geophysical and engineering flows. He analyzed disturbances created in slowly moving subcritical currents whose speed U is less than that of the speed c of long waves, for the particular cases of open-channel water flow, horizontal stably stratified flows, and rotating flows (with the current running parallel to the axis of rotation). In these flows, long wavelength disturbances propagate both upstream and downstream, and the flow pattern along the channel varies with distance and the time since the disturbance was initiated (e.g., a body set into motion). Using the integral properties of the flow at each cross section [as Benjamin & Lighthill (1954) and Whitham (1962) had proposed earlier], and relating these to the strength of the initial disturbance (e.g., the impulse imparted to the body), Benjamin derived relations between upstream and downstream disturbances and conditions under which a "finite disturbance of the fluid always occurs over a continually increasing distance ahead of the body." Because the paper does not provide complete descriptions of any particular flow situation, there are, as Benjamin admits, some ambiguous and uncertain results. But even now the problems of upstream influence are still being overlooked (Hunt et al. 2004).

As in his work on water waves, Benjamin (1986a) also developed the Hamiltonian and symmetry methods to extend his integral approach for the analysis of nonlinear internal waves. By seeking stationary values of the Hamiltonian integrals, and identifying all the complete symmetries of the possible flows defined by this system (Benjamin & Olver 1982), there is a formal "assurance that the complete group (i.e., all possible flow states) has been identified." This approach requires new formal definitions of the classical concepts such as impulse, and of the newer concept of Lyapunov stability. Thence, Benjamin & Bowman (1987) developed a general approach to conservation laws, obviating the ad hoc physical reasoning that has commonly been needed to identify conservation for the simpler case of one-dimensional Hamiltonian systems. They proposed applications of their work to long waves in elastic tubes and on open-channel flows, and generally to gas dynamics.

A similar methodology was applied to the even more complex nonlinear problem of growth and deformation of waves on the interface between two streams of fluid with different density and traveling at different velocities. The linear analysis of this classic problem first derived by Lord Kelvin and Helmholtz, and named after them, had been extended to the weakly nonlinear range by Drazin (1970). Numerical computations using large computers had begun to simulate the evolution of the initial instability into the large-amplitude distortion of the sheet into the rolled-up spiral shapes. But no previous analysis had been proposed to deal with these later stages until Benjamin & Bridges (1997a,b, published posthumously) developed a Hamiltonian formulation for the perturbed total energy E defined in terms of the displacement (η) and a surface function (S) related to the velocity potentials of the flow fields either side of the interface. In the latter paper, the authors apply this theory to the study of nonlinear periodic traveling waves, which can suddenly change as the density and differences between the streams vary. They discover a new connection between Kelvin-Helmholtz billows and the tendency of waves at low amplitude to form into groups, showing again how the Benjamin-Feir sideband instability may have many manifestations.

The final achievement of Benjamin's (1992) research on internal flows was his discovery of yet another form of solitary wave, and the invention of a third nonlinear partial differential equation having interesting mathematical properties (some of which he established). Here there is a parallel horizontal flow of two streams (the upper stream having thickness h larger than that of the lower stream). They have differing density (ρ_1, ρ_2) and are separated by an interface with surface tension T large enough that surface tension waves travel much faster than internal waves (i.e., $\frac{T}{(\rho_1+\rho_2)h} \gg \frac{g(\rho_2-\rho_1)h}{(\rho_1+\rho_2)}$). He introduced the new (Benjamin) equation (for a normalized displacement variable u),

$$u_t + u_x + 2uu_x - \alpha\Lambda u_x - \beta u_{xxx} = 0,$$

where $\alpha, \beta > 0$ and $\beta \gg \alpha$ and Λ is a similar linear differential operator as appears in the Benjamin-Ono equation (1967c). From the general properties of the solution to this equation, explicit solutions not being obtained, it was shown that these solitary waves have oscillatory wave-like forms at their extremities, and their velocities of translation are less than the minimum velocity of infinitesimal waves. These properties contrast with those of solitary waves on a free surface, where the waves have a smooth shape and move faster than infinitesimal waves (with the same wavelength). The Benjamin equation continues to interest mathematicians because, by virtue of its fourth and fifth terms, it combines different dispersive effects of the Benjamin-Ono and the KdV types of wave equations.

WAVES AND INSTABILITIES ON VISCOUS SHEARING FLOWS

As an engineer Benjamin's interest in wavy motions was also directed to flows near boundaries that occur in industrial processes over hydraulic structures and around ships and aircraft. He focused particularly on the general ways in which waves and instabilities growing in shear flows depend on the boundary conditions at the interfaces with adjoining solids and on vibrations and waves excited in solid and fluid regions.

Benjamin was first interested in the onset of waves in a laminar flow of a liquid stream flowing down a slope with angle θ. He computed solutions to the Orr-Sommerfeld linear equations for the growth of small disturbances on the laminar parabolic profile using a very laborious power series method. This led to predictions for the wavelength and wave speed of the most rapidly growing two-dimensional disturbance as the viscosity (or inverse of the Reynolds number Re) of the flow increases, which agreed with the experimental finding that waves become vanishingly small and their wavelength very large as *Re* decreases to a critical level of (5/6 cot θ). These are the familiar linear ripple waves seen on wet roads, windows, etc. The theory showed that for a vertical stream (cot $\theta = 0$), there is no critical value of *Re*, i.e., such waves always occur. However, the Reynolds number of the flow must be finite for finite-amplitude waves to be observed.

Later Benjamin (1964a) extended the analysis to the effects of surface active agents on the onset of these ripple waves. He showed that the elasticity of such agents effectively suppresses the wave growth even when their concentration is very low, so that below a critical Reynolds number no waves exist on liquid sheets flowing down vertical surfaces. He collaborated with Taylor (1968) in analyzing the instability of jet, threads, and sheets of viscous fluids. Benjamin & Mullin (1988) studied a different kind of wavy horizontal interface caused by dragging one layer of a very viscous fluid over another.

Benjamin's (1961) foray into the bigger problems of the stability of shear flows began when he extended his earlier analysis (Benjamin 1957) of waves on a sloping liquid current to provide a "complete theory for the growth of small three dimensional disturbances in a flow of boundary-layer type." He first calculated (as others had) the individual waves traveling at an angle to the flow and then integrated the effect of many such waves using double Fourier integrals to study how a local disturbance evolved. The result agreed nicely with the results of the small experiment, which he used to keep in his room in the engineering department. On a glass plate sloping at an angle about 30° to the horizontal with graph paper attached to its underside, a stream of water ran down into the sink. Short waves remain confined within an elliptical region whose area increases linearly with time while being modulated by longer waves. His prediction that this was a generic feature of unstable shear flows was borne out in many subsequent studies (e.g., Gaster 1968).

From waves on the shearing motions of thin liquid currents, Benjamin (1959) moved on to study larger-scale shearing motions over wavy surfaces rigid of fluid, including air flow over water waves and the related problem of fluid flows (e.g., of gas or liquid) over flexible solid surfaces (see Benjamin 1960, 1963, 1964). Using the classical methods described by Lin (1955) of local power series expansions of the equation near boundaries and near critical layers $y = y_c$ [where the travel speed c_r of the disturbance was equal to that of the velocity $U(y_c)$], he used approximations of the published eigen solutions of the *O-S* equation where possible, but not the methods of matched asymptotic expansions. Benjamin (1959) concluded that in laminar boundary flow over a rigid surface with a low bump (with small slope) the peak shear stress occurred upstream of the top of the bump and the minimum on the downwind side, and that where the flow slowed down the displaced fluid caused the pressure to rise, thus producing a small net drag on the bump (without, as Jeffreys (1925) had supposed, the flow separating). Applying the same method, he then showed why this sheltering mechanism could explain how wind over water waves amplified their growth. [Later research showed how this robust concept is valid even in turbulent flows (Belcher & Hunt 1998).]

A careful reading of Benjamin's review (1964b) suggests that he favored this mechanism over the alternative inviscid critical-layer mechanism for the growth of wind-driven waves proposed by Miles (1957). (More recent research suggests that both mechanisms might operate on groups of wind-driven waves.)

Benjamin (1963) studied the effects of a flexible boundary on the stability of plane parallel flows following the experiments of Kramer (1960), which showed that covering a submerged submarine or torpedo with flexible surfaces could reduce its drag and make it go faster or more silently through the water. As usual he tackled this particular problem by relating it to a wider class of flows, which he defined and analyzed for the first time. He developed a simple classification for the stability of these flows based on the ratio of the speed (c_r) of the most unstable disturbances (or the maximum velocity of the flow U_{mx}) to the speed of elastic waves c_b of the flexible stationary boundary. (*a*) When $c_r < c_b$, the unstable shear-driven waves in the flow have broadly the same form as flow over a rigid surface (where $c_b/c_r \to \infty$), but then they are damped. However, because there is some internal friction within the flexible medium, phase changes are induced and flows can be destabilized. (*b*) When $c_r \sim c_b$, there can be a resonance effect, so that the most unstable waves in the fluid are closely linked to the waves on the flexible surface. (*c*) If the stiffness is very small, and $c_b \ll c_r$, Kelvin-Helmholtz waves on the interface are possible, although this was not completely worked out. These ideas, which he reviewed later 1964b, were soon taken up by other researchers, notably Landahl (1962), by designers of submarine vessels, and by zoologists attempting to understand the surprisingly high speeds that dolphins reach.

MULTIPLE STATES OF SWIRLING FLOWS

Benjamin contributed to our knowledge of swirling flows with novel mathematical approaches and ingenious experiments. He began with an analysis (1962c) of the vortices directed parallel to the air stream that were a distinct feature of air flow over vortex wings, which became the preferred design for supersonic aircraft in the late 1950s. Wind tunnel experiments showed how these steady vortices became unsteady and thickened over a short length at a certain distance from the leading edge of the wing. This vortex-breakdown phenomenon also occurs in swirling flows along pipes. Three previous mechanisms had been proposed for this phenomena, an instability of the boundary layer (particularly on a wing surface), separation, and formation of a standing wave. Benjamin argued that vortex breakdown was essentially a transition between two conjugate states of flows, either of which could be present, given the overall condition of the flow situation. He proposed that these flows are analogous to the hydraulic jump (Benjamin & Lighthill 1954), where the volume flux and flow force (or momentum flux of open-channel flow) enable the possibility of two conjugate states. As with the channel flow, by considering the effects of propagation of waves and dissipation on the energy flux, Benjamin showed that the sudden changes (without an external body force) could only occur where the flow speed changed from being greater than the speed of long waves, i.e., supercritical, to being less, i.e., subcritical. No one previously had seen the analogy with (weak) hydraulic jumps. In 1967 Benjamin (1967a) developed a perturbation analysis for axisymmetric steady waves, which he admitted did not account for the intrinsic nonlinear aspects of the stagnation point in the flow and recirculating stream lines, and the nonaxisymmetric unsteady features of the flow. Other researchers generally concluded that Benjamin's concept provided a framework but not a detailed predictive model (unlike the case of the hydraulic jump) (e.g., Leibovich 1983).

Despite doubts that had been (and continue to be) cast on this concept, Benjamin (1971) went on to develop a general analysis of conjugate flows, with further applications to internal stratified flows, where long waves and breaking waves also tend to have a controlling influence. He applied the methods of functional analysis to generalize the definitions of super- and subcritical flow regimes so that they could be applied in many flow situations. This approach also leads to an instructive relation between these categories and the stationary states of the mean flow force, which is generally a minimum in supercritical flows. Then Benjamin (1972) applied the conjugate flow approach to the analysis of bubbly flows where large-scale wave motions cause dramatic and sudden changes in the concentration of bubbles and in the velocity field.

The areas of fluid mechanics that Benjamin chose to demonstrate the applicability of twentieth century mathematical theory were particularly in the fields of swirling flows and nonlinear waves. In a symposium at Marseille he set out his philosophy (1975): "[T]he tools available in functional analysis can sometimes be extremely expedient in their applications to physical problems, winning ground that is genuinely valuable by the criteria of good science." The "ground" that he won was a better understanding of the variety of the steady and unsteady types of flow pattern that occur in the annulus between rotating cylinders (one or other of which may be stationary). This suggested a framework for the specific nonlinear analyses (e.g., Davey et al. 1968) that have quantified some aspects of the cellular forms that exist above the critical rotation speed or Reynolds number (Coles 1965). He applied the general concept of the index of the possible solutions of the Navier Stokes equations and the calculated forms of the bifurcation in the solutions at the critical Reynolds number. The implication was that using the Leray-Schauder degree theory for certain geometries (and ranges of Reynolds number) where solutions to the Navier Stokes equation are not unique, it would be unstable, although a steady flow could formally exist (Benjamin 1976). In particular, the index theorem shows that it is not possible to have all steady or all unsteady solutions. Therefore, in practice, additional possible flow patterns would be unsteady and might not even be observed. A whole range of possible types of steady and unsteady flow might emerge suddenly at the critical values of *Re*, i.e., when the solutions bifurcate (but almost never into two possible solutions!). In this paper Benjamin (1978a) also considered the bifurcations in the flow patterns in rotating Couette flow as the length (l) of the annulus varied. These are analogous to those found in thermal convection in a box. He made a remarkable prediction about the numbers of cells in relation to l and R, in particular that these could be odd numbered. No one had reported three vortices in an annulus before. This was verified in careful experiments reported in (1978b) and further extended at Oxford with his collaborator, Mullin. As the Reynolds number increases, further bifurcations occur and most observed flows become oscillatory. The experiments of Benjamin & Mullin (1980, 1982) also showed that the flow patterns depend on the history of how they are set up, as the speeds of the inner or outer cylinder are varied. Equally significant is that the history of the flow could also affect the interaction between neighboring steady states, as later experimental and numerical studies by Mullin (1982) and Cliffe (1988) established. The full understanding of such hysteresis phenomena, related as it is to the predictability of flows, is generally seen as one of the outstanding challenges, and mysteries, of fluid mechanics. Undoubtedly, future research in this direction will use the wide range of methods and deep ideas that Benjamin developed and showed how to exploit so effectively.

As a finale, we should remember one of the best "Brooke" stories. He was invited to dinner at 10 Downing Street in the 1980s. Circulating after dinner he met the Prime Minister's husband Denis Thatcher, who asked him who he was. Brooke replied that he was a professor at Oxford, and Denis said, "I should keep that quiet around here, if I were you."

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Contents

INDEXES

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